

Bifurcation analysis of a stochastically driven limit cycle

MAXIMILIAN ENGEL, JEROEN S.W. LAMB AND MARTIN RASMUSSEN

Department of Mathematics, Imperial College London,
South Kensington Campus
London SW7 2AZ, UK,

Correspondence: maximilian.engel13@imperial.ac.uk

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Abstract

We establish the existence of a bifurcation from an attractive random equilibrium to shear-induced chaos for a stochastically driven limit cycle, indicated by a change of sign of the first Lyapunov exponent. This addresses an open problem posed by Kevin Lin and Lai-Sang Young in [20, 31], extending results by Qiudong Wang and Lai-Sang Young [29] on periodically kicked limit cycles to the stochastic context.

Key words. Furstenberg-Khasminskii formula, Lyapunov exponent, Random dynamical system, shear-induced chaos, stochastic bifurcation

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1 Introduction

We consider the following model of a stochastically driven limit cycle

$$\begin{aligned} dy &= -\alpha y dt + \sigma f(\vartheta) \circ dW_t, \\ d\vartheta &= (1 + by) dt, \end{aligned} \tag{1.1}$$

where $(y, \vartheta) \in \mathbb{R} \times \mathbb{S}^1$ are cylindrical amplitude-phase coordinates, and W_t denotes a one-dimensional Brownian motion entering the equation as noise of Stratonovich type. In the absence of noise ($\sigma = 0$), the ODE (1.1) has a globally attracting limit cycle at $y = 0$ if $\alpha > 0$. In the presence of noise ($\sigma \neq 0$), the amplitude is driven by phase-dependent noise. The real parameter b induces shear: if $b \neq 0$, the phase velocity $\frac{d\vartheta}{dt}$ depends on the amplitude y .

The stable limit cycle turns into a random attractor if $\sigma \neq 0$. The main question we address in this paper concerns the nature of this random attractor. The crucial quantity is the sign of the first Lyapunov exponent λ_1 with respect to the invariant measure associated to the random attractor. In essence, λ_1 is the dominant infinitesimal asymptotic expansion rate of almost all trajectories.

To facilitate the analysis, we choose $f : \mathbb{S}^1 \simeq [0, 1) \rightarrow \mathbb{R}$ to be continuous and piecewise linear with constant absolute value of the derivative almost everywhere. The simplest example is given by

$$f(\vartheta) = \begin{cases} \vartheta & \text{if } \vartheta \leq \frac{1}{2}, \\ (1 - \vartheta) & \text{if } \vartheta \geq \frac{1}{2}. \end{cases} \tag{1.2}$$

With this choice of the amplitude-phase coupling we obtain the following bifurcation result. We will show that the result stays robust under perturbations of f that smoothen this function.

Theorem 1.1. *Consider the SDE (1.1) with f given by (1.2). Then for all $\alpha > 0$ and $b \neq 0$, there exist $\sigma_-(\alpha, b) \leq \sigma_0(\alpha, b) \leq \sigma_+(\alpha, b)$ such that the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of the random attractor of (1.1) satisfies*

$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_-(\alpha, b), \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b), \\ > 0 & \text{if } \sigma > \sigma_+(\alpha, b). \end{cases}$$

As long as $b, \sigma \neq 0$, the amplitude variable y can be rescaled so that the shear parameter becomes equal to 1 and the effective noise-amplitude becomes σb . Hence, the above result also holds with the roles of σ and b interchanged. Numerical explorations suggest that $\sigma_-(\alpha, b) = \sigma_0(\alpha, b) = \sigma_+(\alpha, b)$, where $\sigma_0(\alpha, b)$ is a decreasing function of b and an increasing function of α . The latter is illustrated in Figure 1. Figure 1a depicts λ_1 as a function of α and σ for fixed $b = 2$. Figure 1b displays the corresponding areas of positive and negative top Lyapunov exponent in the (σ, α) -parameter space where $\lambda_1 = 0$ along the curve separating the two areas. The curve is increasing away from $(0, 0)$. If $\sigma = 0$, we clearly have $\lambda_1 = 0$ for all $\alpha > 0$. The case $\alpha = 0$ is obviously not of any interest in our model.

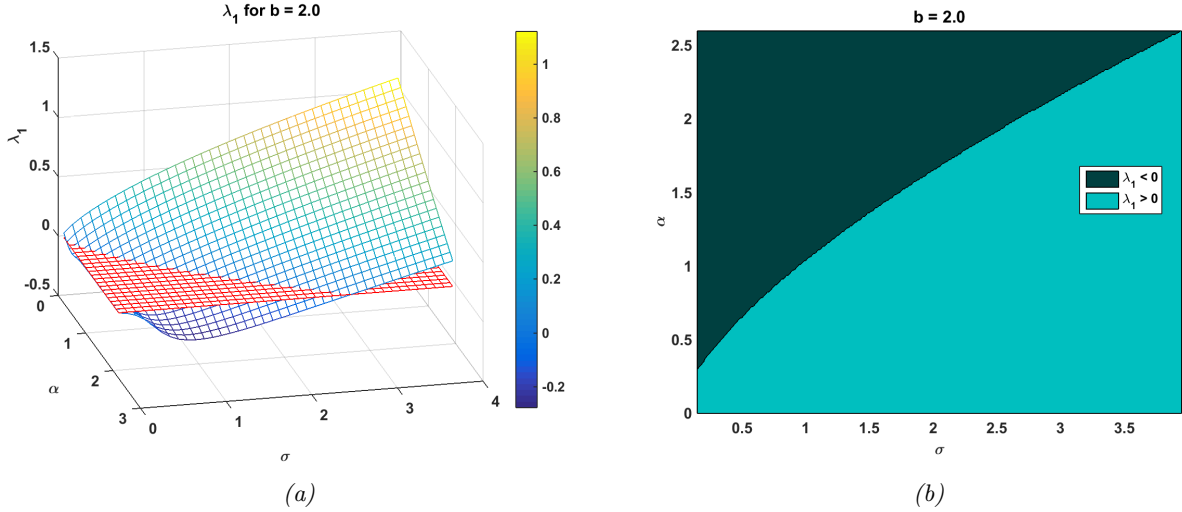


Figure 1: In Figure 1a we depict the top Lyapunov exponent λ_1 , computed using (2.14), as a function of α and σ for fixed $b = 2$. The red mesh demarcates the level $\lambda_1 = 0$. Figure 1b shows the corresponding areas of positive and negative λ_1 in the (σ, α) -parameter space being separated by the curve of $\lambda_1 = 0$ which increases away from $(0, 0)$. The picture doesn't display $\sigma = 0$: in this case, $\lambda_1 = 0$ for all $\alpha > 0$.

If the top Lyapunov exponent of the random attractor is negative, it turns out that the pullback attractor is an *attracting random equilibrium*, i.e. its fibers are singletons almost surely. Properties of random attractors with positive top Lyapunov exponents are not yet well understood, apart from the fact that such attractors are not random equilibria. They are sometimes referred to as *random strange attractors* [18, 30]. Theorem 1.1 thus implies the following dynamical characterization as a dynamical bifurcation:

Corollary 1.2. *If $0 < \sigma < \sigma_-(\alpha, b)$, the random attractor of (1.1) is an attracting random equilibrium. If $\sigma > \sigma_+(\alpha, b)$ the random attractor of system (1.1) is a random strange attractor (and not an attracting random equilibrium).*

Theorem 1.1 and Corollary 1.2 confirm numerical results by Lin & Young [20] for a very similar model. The mechanism, whereby a combination of shear and noise causes stretching and folding leading to a positive Lyapunov exponent, was coined with the term *shear-induced chaos* by Lin & Young. Wang & Young [28, 29] and Ott & Stenlund [23] have demonstrated analytically the validity of this mechanism in the case of periodically kicked limit cycles, including probabilistic characterizations of the dynamics. An analytical proof of shear-induced chaos in the stochastic setting, as presented in this paper, had remained an open problem.

The results of this paper are part of a larger effort to develop a bifurcation theory of random dynamical systems. Despite the relevance of such a theory for many applications of topical interest, the theory of random dynamical systems, and in particular of the bifurcation theory in this context, is still in its infancy. Earlier attempts to develop such a theory (notably by Ludwig Arnold, Peter Baxendale and coworkers [2, 3, 5, 25] in the 1990s) led to notions of so-called *phenomenological* (or "P") bifurcations and *dynamical* (or "D") bifurcations, but there is growing evidence that these paradigms do not suffice to capture the intricacies of bifurcation in random dynamical systems [1, 8, 16, 32]. In the absence of a consensus on useful

characterisations of the dynamics of random systems and bifurcations in this context, much of the current research inevitably focusses on the detailed analysis of relatively elementary examples, to generate insights and guidance towards the further development of a more general theory.

In one-dimensional SDEs, negative Lyapunov exponents and attractive random equilibria prevail [10]. Random strange attractors can only arise in dimension two and higher and up to now, little research has been devoted to such attractors. In contrast, the existence of attractive random equilibria (also referred to as *synchronization*, with reference to the corresponding dynamics of sets of initial conditions) has been studied well, also in higher dimensions [4, 13, 17, 21, 22].

The main technical challenge addressed in this paper is to establish the existence of positive top Lyapunov exponents. Most rigorous results on Lyapunov exponents (and random dynamical systems) are obtained for one-dimensional SDEs, in which case the analysis of Lyapunov exponents significantly simplifies due to the fact that all derivatives commute. It is difficult in general to obtain lower bounds for the top Lyapunov exponent in higher dimensions due to the subadditivity property of matrices, cf. [31]. Thus, the analytical demonstration of positive Lyapunov exponents for noisy systems has been achieved only in certain special cases, like for equilibria [14], simple time-discrete models as in [19] or under special circumstances that allow for the use of stochastic averaging [6, 7]. In our setting, the choice of f in (1.2) is crucial to establish rigorous lower bounds on the top Lyapunov exponent λ_1 .

Another prototypical open problem in dimension two is the *stochastic Hopf bifurcation*, concerning the characterisation of dynamics and bifurcations in parametrized families of SDEs that in the deterministic (noise-free) limit display a *Hopf bifurcation*. The (deterministic) Hopf bifurcation is an oscillatory instability where by the variation of a model parameter, an asymptotically stable equilibrium loses stability under the emission of a small attracting limit cycle. Many studies in this context have considered the Duffing–Van der Pol oscillator with multiplicative white noise [3, 25, 26]. Few rigorous results have been obtained and most studies only led to conjectures based on numerical observations [15, 30]. Indeed, numerical studies suggest that the mechanism of shear-induced chaos is at play also in stochastic Hopf bifurcations, but while analytical proofs of parameter regimes with negative top Lyapunov exponents are within reach [11], until now, there are no rigorous results concerning the existence of parameter regimes with positive top Lyapunov exponents in this context. The results of this paper may well be relevant to shed more light on this problem.

The remainder of the paper is organized as follows. Section 2 provides the analysis of Lyapunov exponents for our model: Subsection 2.1 introduces the model on the cylinder within the framework of random dynamical systems and establishes the necessary theoretical concepts. Subsection 2.2 introduces the Furstenberg–Khasminskii formula for the top Lyapunov exponent and in Subsection 2.3, we derive a formula for the top Lyapunov exponent λ_1 in the case of the specific choice of the diffusion map f . The main result concerning the change of sign of λ_1 is proven in Section 3 and its consequences are discussed. We give illustrations of λ_1 in dependence on the parameters, confirm a scaling conjecture by Lin and Young and explain why the results stay robust under smooth perturbations of f . We conclude with a short summary of the results and open problems in Section 4.

2 Analysis of the top Lyapunov exponent

2.1 Lyapunov exponents for random dynamical systems

Consider the stochastic differential equation of Stratonovich type (1.1). We assume that $f : [0, 1] \rightarrow \mathbb{R}$ is a Lipschitz continuous function with $f(0) = f(1)$, and the three parameters fulfill $\alpha > 0$, $\sigma \geq 0$ and $b \geq 0$. Note that the equation reads the same in Itô form according to the Itô–Stratonovich conversion formula (cf. Appendix).

We investigate this model in the framework of random dynamical systems. A one-sided continuous-time random dynamical system [2, Definition 1.1.2] consists of the following:

- (i) A model of the noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, formalized as a measurable flow $(\theta_t)_{t \in \mathbb{R}}$ of \mathbb{P} -preserving transformations $\theta_t : \Omega \rightarrow \Omega$, i.e. satisfying $\theta_0 = \text{id}$, $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s \in \mathbb{R}$, $\theta_t \mathbb{P} = \mathbb{P}$ and $(t, \omega) \mapsto \theta_t \omega$ measurable.
- (ii) A model of the system perturbed by noise formalized as a *cocycle* φ over θ of mappings of the metric

state space X , i.e. φ is a $\mathcal{B}(\mathbb{R}_0^+) \otimes \mathcal{F} \otimes \mathcal{B}(X)$ -measurable mapping

$$\varphi : \mathbb{R}_0^+ \times \Omega \times X \rightarrow X, \quad (t, \omega, x) \mapsto \varphi(t, \omega)x,$$

such that $(t, x) \mapsto \varphi(t, \omega)x$ is continuous for every $\omega \in \Omega$ and which satisfies

$$\varphi(0, \omega) = \text{id} \quad \text{and} \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \quad \text{for all } \omega \in \Omega \text{ and } t, s \in \mathbb{R}_0^+.$$

Since in our case, the drift and diffusion coefficients are Lipschitz continuous and satisfy linear growth conditions, the SDE generates a continuous random dynamical system (φ, θ) : the probability space Ω is $C_0(\mathbb{R}, \mathbb{R}) = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, \mathcal{F} is the Borel σ -algebra and \mathbb{P} is the two-sided Wiener measure. \mathcal{F}_s^t is the σ -algebra generated by $\xi_u - \xi_v$ for $-\infty \leq s \leq v \leq u \leq t \leq \infty$, where $\xi_s : \Omega \rightarrow \mathbb{R}$ is defined as $\xi_s(\omega) = \omega(s)$, and θ_t is the ergodic shift

$$(\theta_t \omega)(s) = \omega(s + t) - \omega(t).$$

As we are working with dissipative models, the random system will be forward complete, i.e. $\varphi(t, \omega)$ is defined for all $t \geq 0$.

The random dynamical system (θ, φ) induced by (1.1) is also a skew product flow $\Theta = (\theta, \varphi)$, which is a measurable dynamical system on the extended phase space $\Omega \times X$ (see Appendix). The skew product flow Θ possesses an ergodic invariant Markov measure μ which is associated to the unique invariant measure (also called stationary measure) for the corresponding Markov semigroup (see Appendix). Their existence follows from the same considerations as in [20]. Furthermore, the random dynamical system induced by (1.1) has a random attractor $\{A(\omega)\}_{\omega \in \Omega}$ (see Appendix for a formal definition), since the diffusion is bounded and the drift satisfies a dissipative one-sided Lipschitz condition [24, Chapter 14.3]. The disintegrations μ_ω of the ergodic invariant measure μ are supported on the fibers $A(\omega)$ of this random attractor.

Fundamental for stochastic bifurcation theory is Oseledets' Multiplicative Ergodic Theorem, which implies the existence of Lyapunov exponents describing stability properties of a differentiable random dynamical system. The random dynamical system (θ, φ) is called C^k if $\varphi(t, \omega) \in C^k$ for all $t \in \mathbb{R}_0^+$ and $\omega \in \Omega$. In the situation of the Stratonovich SDE

$$dX_t = f_0(X_t)dt + \sum_{j=1}^m f_j(X_t) \circ dW_t^j$$

on a smooth manifold X , the Jacobian $D\varphi(t, \omega, x)$ with respect to the third variable of the cocycle $\varphi(t, \omega)x$ is a linear cocycle over the skew product flow $\Theta = (\theta, \varphi)$. The Jacobian $D\varphi(t, \omega, x)$ applied to an initial condition $v_0 \in T_x X$ solves uniquely the variational equation on $T_x X \cong \mathbb{R}^d$, given by

$$dv = Df_0(\varphi(t, \omega)x)v dt + \sum_{j=1}^m Df_j(\varphi(t, \omega)x)v \circ dW_t^j, \quad \text{where } v \in T_x X. \quad (2.1)$$

The Jacobian $D\varphi(t, \omega, x)$ satisfies Liouville's equation

$$\det D\varphi(t, \omega, x) = \exp \left(\int_0^t \text{trace } Df_0(\varphi(s, \omega)x) ds + \sum_{j=1}^m \int_0^t \text{trace } Df_j(\varphi(s, \omega)x) \circ dW_s^j \right). \quad (2.2)$$

Suppose the one-sided C^1 -random dynamical system (φ, θ) has an ergodic invariant measure ν and satisfies the integrability condition

$$\sup_{0 \leq t \leq 1} \log^+ \|D\varphi(t, \omega, x)\| \in L^1(\nu).$$

Then the Multiplicative Ergodic Theorem for differentiable random dynamical systems [2, Theorem 3.4.1, Theorem 4.2.6] guarantees the existence of a Θ -forward invariant set $\Delta \subset \Omega \times X$ with $\nu(\Delta) = 1$ and the Lyapunov exponents $\lambda_1 > \dots > \lambda_p$ with respect to ν . Denote by d_i the multiplicity of the Lyapunov exponent λ_i , which implies $\sum_{i=1}^p d_i = d$. The tangent space $T_x X \cong \mathbb{R}^d$ admits a filtration

$$\mathbb{R}^d = V_1(\omega, x) \supsetneq V_2(\omega, x) \supsetneq \dots \supsetneq V_p(\omega, x) \supsetneq V_{p+1}(\omega, x) = \{0\},$$

such that for all $0 \neq v \in T_x X \cong \mathbb{R}^d$, the Lyapunov exponent $\lambda(\omega, x, v)$ defined by

$$\lambda(\omega, x, v) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|D\varphi(t, \omega, x)v\|$$

exists and

$$\lambda(\omega, x, v) = \lambda_i \iff v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \dots, p\}.$$

Note that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \det D\varphi(t, \omega, x) = \sum_{i=1}^p d_i \lambda_i. \quad (2.3)$$

2.2 The Furstenberg–Khasminskii formula

In the following, we calculate the top Lyapunov exponent λ_1 for the random dynamical system induced by (1.1). We consider the corresponding variational equation describing the flow on the tangent space $T_x(\mathbb{R} \times \mathbb{S}^1) \cong \mathbb{R}^2$ along trajectories of (1.1). We assume that $f \in C^1$ for now, although we consider a more general case later. The variational equation (2.1) reads as

$$dv = \underbrace{\begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix}}_{=:A} v dt + \underbrace{\begin{pmatrix} 0 & \sigma f'(\vartheta) \\ 0 & 0 \end{pmatrix}}_{=:B} v \circ dW_t. \quad (2.4)$$

Note that we omit the (t, ω) -dependence of ϑ and B . Because of the linearity of (2.4), we introduce the change of variables $r = \|v\|$ and $s = v/r$, so that s lies on the unit circle. Its dynamics are given by

$$\begin{aligned} ds &= (As - \langle s, As \rangle s) dt + (Bs - \langle s, Bs \rangle s) \circ dW_t \\ &= \begin{pmatrix} -\alpha s_1 - s_1(-\alpha s_1^2 + b s_1 s_2) \\ b s_1 - s_2(-\alpha s_1^2 + b s_1 s_2) \end{pmatrix} dt + \begin{pmatrix} \sigma f'(\vartheta) s_2 - s_1 \sigma f'(\vartheta) s_1 s_2 \\ -s_2 \sigma f'(\vartheta) s_1 s_2 \end{pmatrix} \circ dW_t. \end{aligned}$$

The Furstenberg–Khasminskii formula for the top Lyapunov exponent [14] is given by

$$\lambda_1 = \int_{\mathbb{R}} \int_{[0,1]} \int_{\mathbb{S}^1} (h_A(s) + k_B(s)) \rho(ds, d\vartheta, dy), \quad (2.5)$$

where ρ is the joint invariant measure for the diffusion s on the unit circle and the processes ϑ and y induced by (1.1); the functions h_A and k_B are given by

$$\begin{aligned} h_A(s) &= \langle s, As \rangle = -\alpha s_1^2 + b s_1 s_2, \\ k_B(s) &= \frac{1}{2} \langle (B + B^*)s, Bs \rangle - \langle s, Bs \rangle^2 = \frac{1}{2} \sigma^2 f'(\vartheta)^2 s_2^2 - \sigma^2 f'(\vartheta)^2 s_1^2 s_2^2. \end{aligned}$$

Similarly to the calculations in [14], we change variables to $s = (\cos \phi, \sin \phi)$. Note that the functions h_A and k_B are π -periodic, which implies that the formula (2.5) for the top Lyapunov exponent reads as

$$\lambda_1 = \int_{\mathbb{R} \times [0,1] \times [0,\pi]} \left(-\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2} \sigma^2 f'(\vartheta)^2 \sin^2 \phi (1 - 2 \cos^2 \phi) \right) \tilde{\rho}(d\phi, d\vartheta, dy), \quad (2.6)$$

where $\tilde{\rho}$ denotes the corresponding image measure of ρ . The SDE determining the dynamics of $\phi \in [0, \pi)$ reads as

$$d\phi = -\frac{1}{\sin \phi} ds_1 = (\alpha \cos \phi \sin \phi + b \cos^2 \phi) dt - \sigma f'(\vartheta) \sin^2 \phi \circ dW_t, \quad (2.7)$$

where we denote

$$c(\phi, \vartheta) = \sigma f'(\vartheta) \sin^2 \phi \quad \text{and} \quad d(\phi) = \alpha \cos \phi \sin \phi + b \cos^2 \phi. \quad (2.8)$$

In the Fokker–Planck equation for ϕ , the dependence on ϑ is restricted to $f'(\vartheta)^2$ (see Appendix), and in addition to that, the integrand of (2.6) only depends on ϕ and not on ϑ and y if $f'(\vartheta)^2$ is constant. This means that the calculation of λ_1 becomes much simpler if $f'(\vartheta)^2$ is constant, an observation that we exploit in the following.

2.3 Explicit formula for the top Lyapunov exponent

We continue the analysis of the top Lyapunov exponent under the assumption that $f : [0, 1] \rightarrow \mathbb{R}$ is given by (1.2). Importantly, $f'(\vartheta)^2$ is constant in this special case and our results hold in fact for every continuous and piecewise linear f with constant absolute value of the derivative almost everywhere.

The map is not differentiable at $\frac{1}{2}$, and we verify that does not cause any problems. We need the following results to justify the variational equation defining $D\varphi$:

Lemma 2.1. *Let $W : \mathbb{R}_0^+ \times \Omega \rightarrow \mathbb{R}$ denote the canonical real-valued Wiener process, and let $X : \mathbb{R}_0^+ \times \Omega \rightarrow [0, 1]$ be a stochastic process adapted to the natural filtration of the Wiener process. Furthermore, suppose there exists a measurable set $A \subset [0, 1]$ such that*

$$\mathbb{P}(\{\omega \in \Omega : \int_0^t \mathbf{1}_{\{X_u \in A\}} du = 0\}) = 1 \quad \text{for all } t > 0, \quad (2.9)$$

i.e. A is visited only on a measure zero set with full probability. Consider a measurable function $g : [0, 1] \rightarrow [0, 1]$ such that $g = 0$ on $[0, 1] \setminus A$. Then

$$\int_0^t g(X_u) dW_u = 0 \quad \text{almost surely for all } t > 0.$$

Proof. The statement follows directly from Itô's isometry

$$\mathbb{E} \left[\left(\int_0^t g(X_u) dW_u \right)^2 \right] = \mathbb{E} \left[\int_0^t g(X_u)^2 du \right] = \mathbb{E} \left[\int_0^t \left(g(X_u)^2 \mathbf{1}_{\{X_u \in A\}} + g(X_u)^2 \mathbf{1}_{\{X_u \in [0, 1] \setminus A\}} \right) du \right] = 0,$$

where the last equality follows immediately from (2.9) and $g = 0$ on $[0, 1] \setminus A$. We conclude

$$\left(\int_0^t g(X_u) dW_u \right)^2 = 0 \quad \text{almost surely}$$

due to nonnegativity, and the claim follows. \square

Proposition 2.2. *Let f' denote the weak derivative of f as given by (1.2). Then the choice of representative of f' by determining $f'(\frac{1}{2})$ does not affect the solution to the variational equation (2.4).*

Proof. First, we show that

$$\mathbb{P}(\{\omega \in \Omega : \int_0^t \mathbf{1}_{\{\vartheta_u = 1/2\}} du = 0\}) = 1 \quad \text{for all } t > 0$$

by assuming the contrary to obtain a contradiction. As ϑ is a continuously differentiable process, this implies that $\vartheta_u = \frac{1}{2}$ for $u \in [t^*, t^* + \varepsilon]$ for some $t^* \in (0, t)$ and $\varepsilon > 0$ with positive probability. This leads to $y(u) = -\frac{1}{b} \bmod 1$ for $u \in (t^*, t^* + \varepsilon)$ with positive probability. However, this implies that the continuous process y_u for $u \in (t^*, t^* + \varepsilon)$ given by

$$dy = -\alpha y du + \sigma dW_u$$

is constant with positive probability. This contradicts its definition as an Ornstein–Uhlenbeck process.

Let $f'_1 = f'_2 = f'$ on $[0, 1] \setminus \{\frac{1}{2}\}$ and assign arbitrary values at $\frac{1}{2}$. Define

$$\begin{aligned} dv &= \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v dt + \begin{pmatrix} 0 & \sigma f'_1(\vartheta) \\ 0 & 0 \end{pmatrix} v \circ dW_t, \\ dw &= \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} w dt + \begin{pmatrix} 0 & \sigma f'_2(\vartheta) \\ 0 & 0 \end{pmatrix} w \circ dW_t. \end{aligned}$$

We apply Lemma 2.1 by choosing $X_u = \vartheta_u$ and $g(\vartheta_u) = f'_1(\vartheta_u) - f'_2(\vartheta_u)$ to conclude that

$$\int_0^t f'_1(\vartheta_u) dW_u = \int_0^t f'_2(\vartheta_u) dW_u \quad \text{almost surely.}$$

As we do not have an Itô–Stratonovich correction in this case, we can infer that $v_t = w_t$ almost surely for all $t > 0$. \square

We view f' in the weak sense, disregarding the point $\frac{1}{2}$, and we define $f'(\vartheta) = \text{sign}(\frac{1}{2} - \vartheta)$, where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{if } x < 0. \end{cases}$$

By Proposition 2.2, $D\varphi(t, \omega, x)$ does not depend on the choice of $f'(\frac{1}{2})$, so the variational equation (2.4) becomes

$$dv = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v dt + \begin{pmatrix} 0 & \sigma \text{sign}(\frac{1}{2} - \vartheta_t) \\ 0 & 0 \end{pmatrix} v \circ dW_t. \quad (2.10)$$

We derive the following formula for the first Lyapunov exponent in this case:

Proposition 2.3. *The top Lyapunov exponent of system (1.1) with f as defined in (1.2) is given by*

$$\lambda_1 = \int_0^\pi q(\phi) p(\phi) d\phi, \quad (2.11)$$

where $q(\phi) := -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2} \sigma^2 (1 - 2 \cos^2 \phi) \sin^2 \phi$, and $p(\phi)$ is the solution of the stationary Fokker-Planck equation $\mathcal{L}^* p = 0$. \mathcal{L}^* is the formal L^2 -adjoint of the generator \mathcal{L} , which is given by

$$\mathcal{L}g(\phi) = \left(d(\phi) + \frac{1}{2} \tilde{c}(\phi) \tilde{c}'(\phi) \right) g'(\phi) + \frac{1}{2} \tilde{c}^2(\phi) g''(\phi), \quad (2.12)$$

where $d = d(\phi)$ is defined as in (2.8), and $\tilde{c}(\phi) := \sigma \sin^2 \phi$.

Proof. Note that in our special case, the function c from (2.8) reads as

$$c(\phi, \vartheta) = \sigma \text{sign}(\frac{1}{2} - \vartheta) \sin^2 \phi,$$

which implies that both $c(\phi, \vartheta) c'(\phi, \vartheta)$ and $c^2(\phi, \vartheta)$ do not depend on ϑ and read as $\tilde{c} \tilde{c}'$ and \tilde{c}^2 , respectively. Consider the SDE for the process $\phi(t)$ in Itô form

$$d\phi = r(\phi) dt + c(\phi, \vartheta) dW_t,$$

where

$$r(\phi) = d(\phi) + \frac{1}{2} c(\phi, \vartheta) c'(\phi, \vartheta) = d(\phi) + \frac{1}{2} \tilde{c}(\phi) \tilde{c}'(\phi).$$

As the coefficients of the SDE are smooth in ϕ , we consider the kinetic equation for the probability density function of the process $\phi(t)$ (cf. [27])

$$\frac{\partial p(\psi, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial \psi^n} [a_n(\psi, t) p(\psi, t)],$$

where

$$a_n(\psi, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \mathbb{E}[(\phi(t + \Delta t) - \phi(t))^n | \phi(t) = \psi] \quad \text{for all } n \in \mathbb{N}.$$

Pick Δt small, denote $\Delta W_t = W(t + \Delta) - W(t)$ and recall that $\mathbb{E}[\Delta W_t] = 0$ and $\mathbb{E}[(\Delta W_t)^2] = \Delta t$. Observe that

$$\phi(t + \Delta t) - \phi(t) = r(\phi(t)) \Delta t + c(\phi(t), \vartheta(t)) \Delta W_t + o(\Delta t),$$

and

$$\begin{aligned} (\phi(t + \Delta t) - \phi(t))^2 &= r^2(\phi(t)) (\Delta t)^2 + c^2(\phi(t), \vartheta(t)) (\Delta W_t)^2 \\ &\quad + 2r(\phi(t)) c(\phi(t), \vartheta(t)) \Delta W_t \Delta t + o(\Delta t). \end{aligned}$$

Since ΔW_t is independent from $\phi(t)$ and $\vartheta(t)$, we obtain that

$$a_1(\psi, t) = r(\psi) \quad \text{and} \quad a_2(\psi, t) = \tilde{c}^2(\psi).$$

We can see immediately from above that $a_n(\psi, t) = 0$ for $n \geq 3$. This proves (2.12), and (2.11) follows from (2.6). \square

In this case, the stationary Fokker–Planck equation reduces to a linear nonautonomous ordinary differential equation for $p = p(\phi)$ defined on $[0, \pi)$ with periodic boundary conditions:

$$-\left(\frac{1}{2}\tilde{c}^2 p\right)' + \left(d + \frac{1}{2}\tilde{c}\tilde{c}'\right)p = \kappa,$$

where the constant κ has to be determined from the boundary and the normalization condition. The ordinary differential equations is given in explicit form as

$$\begin{aligned} p' &= \left(\frac{2d(\phi)}{\tilde{c}^2(\phi)} - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right)p + \frac{2\kappa}{\tilde{c}^2(\phi)} \\ &= \left(2\frac{\alpha}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-1}\phi + 2\frac{b}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-2}\phi - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right)p + \frac{2\kappa}{\tilde{c}^2(\phi)}. \end{aligned}$$

The solution of this equation follows from the variation of constants formula, and is given by

$$p(\phi) = \frac{G(\phi) \int_{\phi}^{\pi} \frac{2}{\tilde{c}^2(\psi)G(\psi)} d\psi}{\int_0^{\pi} G(\phi) \int_{\phi}^{\pi} \frac{2}{\tilde{c}^2(\psi)G(\psi)} d\psi d\phi}, \quad (2.13)$$

where

$$G(\phi) = \frac{1}{\tilde{c}(\phi)} \exp\left(-\frac{1}{\sigma^2} \left[\alpha \tan^{-2}\phi + \frac{2}{3}b \tan^{-3}\phi\right]\right).$$

The derivation of a closed formula for λ_1 and λ_2 using (2.13) for the stationary density of the process ϕ_t closely follows Imkeller and Lederer [14].

Theorem 2.4. *Consider the stochastic differential equation (1.1), where the function f is of the form (1.2). Then the two Lyapunov exponents are given by*

$$\lambda_1 = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2} \int_0^{\infty} v m_{\sigma,b,\alpha}(v) dv, \quad (2.14)$$

$$\lambda_2 = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2} \int_0^{\infty} v m_{\sigma,b,\alpha}(v) dv. \quad (2.15)$$

where

$$m_{\sigma,b,\alpha}(v) = \frac{\frac{1}{\sqrt{v}} \exp\left(-\frac{\sigma^4 b^4}{6}v^3 + \frac{\alpha^2}{2}v\right)}{\int_0^{\infty} \frac{1}{\sqrt{u}} \exp\left(-\frac{\sigma^4 b^4}{6}u^3 + \frac{\alpha^2}{2}u\right) du}. \quad (2.16)$$

Proof. We define the function $g : [0, \phi) \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\begin{aligned} g(\phi) &:= -\log \sin \phi \quad \text{for all } \phi \in (0, \pi), \\ g(0) &:= \infty, \end{aligned}$$

and apply this function formally to the generator as given in (2.12):

$$\begin{aligned} \mathcal{L}g(\phi) &= (\alpha \cos \phi \sin \phi + b \cos^2 \phi + \sigma^2 \sin^3 \phi \cos \phi) (-\tan^{-1} \phi) + \frac{1}{2}\sigma^2 \sin^4 \phi \frac{1}{\sin^2 \phi} \\ &= -b \tan^{-1} \phi + q(\phi). \end{aligned}$$

This can be made precise by choosing suitable C^∞ -functions to approximate g . Observe that

$$0 = \int_0^{\pi} g \mathcal{L}^* p d\phi = \int_0^{\pi} \mathcal{L}g p d\phi = \int_0^{\pi} (-b \tan^{-1} + q) p d\phi,$$

and we conclude that

$$\lambda_1 = b \int_0^{\pi} \tan^{-1}(\phi) p(\phi) d\phi.$$

Working with expression (2.13), we conduct a change of variables $s = \tan^{-1} \phi$ and $t = \tan^{-1} \psi$ which leads to

$$\lambda_1 = b \frac{\int_{-\infty}^{\infty} \int_{-\infty}^s s \exp \left(-\frac{1}{\sigma^2} [\alpha(s^2 - t^2) + \frac{2}{3}b(s^3 - t^3)] \right) dt ds}{\int_{-\infty}^{\infty} \int_{-\infty}^s \exp \left(-\frac{1}{\sigma^2} [\alpha(s^2 - t^2) + \frac{2}{3}b(s^3 - t^3)] \right) dt ds}. \quad (2.17)$$

We introduce a new variable $u = s - t$, which implies that $u \in (0, \infty)$. We observe

$$\begin{aligned} \alpha s^2 - \alpha(s - u)^2 + \frac{2}{3}bs^3 - \frac{2}{3}b(s - u)^3 &= -\alpha u^2 + 2\alpha su + 2bus^2 - 2bu^2s + \frac{2}{3}bu^3 \\ &= 2bu \left(s - \frac{u - \alpha/b}{2} \right)^2 + \frac{b}{6}u^3 - \frac{1}{2}u \frac{\alpha^2}{b}. \end{aligned}$$

Using this expression, we modify (2.17) and obtain

$$\begin{aligned} \lambda_1 &= b \frac{\int_0^{\infty} \int_{-\infty}^{\infty} s \exp \left(-\frac{2u}{\sigma^2} \left(s - \frac{u - \alpha/b}{2} \right)^2 \right) ds \exp \left(-\frac{b}{6\sigma^2}u^3 + \frac{1}{2}u \frac{\alpha^2}{\sigma^2 b} \right) du}{\int_0^{\infty} \int_{-\infty}^{\infty} \exp \left(-\frac{2bu}{\sigma^2} \left(s - \frac{u - \alpha/b}{2} \right)^2 \right) ds \exp \left(-\frac{b}{6\sigma^2}u^3 + \frac{1}{2}u \frac{\alpha^2}{\sigma^2 b} \right) du} \\ &= b \frac{\int_0^{\infty} \frac{1}{\sqrt{u}} \frac{u - \alpha/b}{2} \exp \left(-\frac{b}{6\sigma^2}u^3 + \frac{1}{2}u \frac{\alpha^2}{\sigma^2 b} \right) du}{\int_0^{\infty} \frac{1}{\sqrt{u}} \exp \left(-\frac{b}{6\sigma^2}u^3 + \frac{1}{2}u \frac{\alpha^2}{\sigma^2 b} \right) du} \\ &= -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2} \frac{\int_0^{\infty} \frac{1}{\sqrt{v}} v \exp \left(-\frac{\sigma^4 b^4}{6}v^3 + \frac{\alpha^2}{2}v \right) dv}{\int_0^{\infty} \frac{1}{\sqrt{v}} \exp \left(-\frac{\sigma^4 b^4}{6}v^3 + \frac{\alpha^2}{2}v \right) dv}, \end{aligned}$$

where we have done another change of variables $v = u/(b\sigma^2)$ in the last equality, and we used well-known properties of the normal distribution. Hence, we write

$$\lambda_1 = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2} \int_0^{\infty} v m_{\sigma,b,\alpha}(v) dv,$$

where $m_{\sigma,b,\alpha}(v)$ is given as in (2.16). From Liouville's formula, we obtain that $\lambda_1 + \lambda_2 = -\alpha$, and this means that

$$\lambda_2 = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2} \int_0^{\infty} v m_{\sigma,b,\alpha}(v) dv.$$

This finishes the proof of this theorem. \square

3 The bifurcation result

3.1 Bifurcation from negative to positive Lyapunov exponent

We now use Theorem 2.4 to prove Theorem 1.1, which asserts that there is a bifurcation from negative to positive Lyapunov exponent for the stochastic differential equation (1.1). We will show in Subsection 3.2 that the result stays robust under perturbations of f that smoothen this function.

Theorem 1.1. *Consider the SDE (1.1) with f given by (1.2). Then for all $\alpha > 0$ and $b \neq 0$, there exist $\sigma_-(\alpha, b) \leq \sigma_0(\alpha, b) \leq \sigma_+(\alpha, b)$ such that the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of the random attractor of (1.1) satisfies*

$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_-(\alpha, b), \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b), \\ > 0 & \text{if } \sigma > \sigma_+(\alpha, b). \end{cases}$$

Proof. We fix $\alpha > 0$ and $b \neq 0$, and define $\gamma := b^2 \sigma^2$. We obtain from (2.14) that

$$\lambda_1 = -\frac{\alpha}{2} + \frac{\gamma}{2} \int_0^\infty v m_{\gamma, \alpha}(v) dv,$$

where

$$m_{\gamma, \alpha}(v) = \frac{\frac{1}{\sqrt{v}} \exp\left(-\frac{\gamma^2}{6} v^3 + \frac{\alpha^2}{2} v\right)}{\int_0^\infty \frac{1}{\sqrt{w}} \exp\left(-\frac{\gamma^2}{6} w^3 + \frac{\alpha^2}{2} w\right) dw}.$$

We introduce another change of variables $v = \frac{\alpha}{\gamma} u$ and obtain

$$\lambda_1 = \frac{\alpha}{2} \left(\int_0^\infty u \tilde{m}_{\gamma, \alpha}(u) du - 1 \right), \quad (3.1)$$

where

$$\tilde{m}_{\gamma, \alpha}(u) = \frac{\frac{1}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right)}{\int_0^\infty \frac{1}{\sqrt{w}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} w^3 - \frac{1}{2} w\right]\right) dw}.$$

Define

$$I_1^\gamma(u) := \frac{u}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right) \quad \text{and} \quad I_2^\gamma(u) := \frac{1}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right).$$

We observe that because of (3.1), $\lambda_1 > 0$ is equivalent to

$$\int_0^\infty I_1^\gamma(u) du > \int_0^\infty I_2^\gamma(u) du. \quad (3.2)$$

Observe that for $t > 3$ (using dominated convergence),

$$\lim_{\gamma \rightarrow \infty} \int_0^t I_1^\gamma(u) du = \int_0^t \sqrt{u} du = \frac{2}{3} t^{\frac{3}{2}} > 2\sqrt{t} = \int_0^t u^{-\frac{1}{2}} du = \lim_{\gamma \rightarrow \infty} \int_0^t I_2^\gamma(u) du$$

and

$$\int_t^\infty I_1^\gamma(u) du > \int_t^\infty I_2^\gamma(u) du \quad \text{for all } \gamma > 0.$$

This implies that there exists a $\gamma_+ > 0$ such that (3.2) holds for all $\gamma > \gamma_+$, and hence, $\lambda_1 > 0$ for all $\gamma > \gamma_+$. Since $\gamma = b^2 \sigma^2$ is monotonically increasing in σ , the existence of $\sigma_+(\alpha, b)$ follows.

Similarly, we show that there is a $\gamma_- > 0$ such that $\lambda_1 < 0$ for all $\gamma < \gamma_-$. Note that λ_1 is negative if

$$\int_0^\infty (I_2^\gamma(u) - I_1^\gamma(u)) du > 0,$$

because of (3.1). For any given $\varepsilon > 0$ we can choose γ small enough such that

$$0 < \int_{\sqrt{3}}^\infty (I_1^\gamma(u) - I_2^\gamma(u)) du < \varepsilon.$$

It is further easy to observe that

$$\begin{aligned} \int_{2-\sqrt{3}}^{\sqrt{3}} (I_2^\gamma(u) - I_1^\gamma(u)) du &= \int_{2-\sqrt{3}}^1 (I_2^\gamma(u) - I_1^\gamma(u)) du - \int_1^{\sqrt{3}} (I_1^\gamma(u) - I_2^\gamma(u)) du = \\ &= \int_{2-\sqrt{3}}^1 \frac{1-u}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right) du - \int_1^{\sqrt{3}} \frac{u-1}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right) du > 0, \end{aligned}$$

and that

$$\int_0^{2-\sqrt{3}} (I_2^\gamma(u) - I_1^\gamma(u)) du = \int_0^{2-\sqrt{3}} \frac{1-u}{\sqrt{u}} \exp\left(-\frac{\alpha^3}{\gamma} \left[\frac{1}{6} u^3 - \frac{1}{2} u\right]\right) du > \varepsilon \quad \text{if } \gamma \text{ is small enough.}$$

Hence, we can deduce that λ_1 is negative for all $\gamma < \gamma_-$ for some $\gamma_- > 0$. Since $\gamma = b^2\sigma^2$ is monotonically increasing in σ , the existence of $\sigma_-(\alpha, b)$ follows.

We have established that there are $\sigma_+(\alpha, b) \geq \sigma_-(\alpha, b)$ such that $\lambda_1(\alpha, b, \sigma) > 0$ for $\sigma > \sigma_+(\alpha, b)$ and $\lambda_1(\alpha, b, \sigma) < 0$ for $\sigma < \sigma_-(\alpha, b)$. By continuity of λ_1 in σ , there exists a σ_0 such that $\lambda_1(\alpha, b, \sigma) = 0$. \square

Remark 3.1. As explained in the Introduction, the same result holds if we interchange the roles of σ and b . This can be seen also directly from the proof above.

For proving the existence of a random equilibrium when $\lambda_1 < 0$, we deploy the following theorem that summarizes [13, Section 2.1].

Theorem 3.2. *Assume that the random dynamical system (θ, φ) on a Polish space X has a \mathcal{F}_∞^0 -measurable weak random attractor $\{A(\omega)\}_{\omega \in \Omega}$ and is*

a) *asymptotically stable on a non-empty open set $U \subset X$, i.e. there exists a sequence $t_n \rightarrow \infty$ such that*

$$\mathbb{P}(\omega \in \Omega : \lim_{n \rightarrow \infty} \text{diam}(\varphi(t_n, \omega, U)) = 0) > 0,$$

b) *swift transitive, i.e. for every ball $B(x, r) \subset X$ and point $y \in X$, there exists a $t > 0$ such that*

$$\mathbb{P}(\omega \in \Omega : \varphi(t, \omega, B(x, r)) \subset B(y, 2r)) > 0,$$

c) *contracting on large sets, i.e. for every $R > 0$ there exists a ball $B(y, R) \subset X$ and a time $t > 0$ such that*

$$\mathbb{P}(\omega \in \Omega : \text{diam}(\varphi(t, \omega, B(y, R))) \leq \frac{R}{4}) > 0.$$

Then $\{A(\omega)\}_{\omega \in \Omega}$ is a singleton \mathbb{P} -almost surely, i.e. the random dynamical system admits synchronization.

We are prepared to deduce Corollary 1.2.

Proof of Corollary 1.2. The fact, that the random attractor $\{A(\omega)\}_{\omega \in \Omega}$ of (1.1) is a singleton almost surely if $\lambda_1 < 0$, follows from Theorem 3.2: by [13, Corollary 3.4], asymptotic stability follows from the negative top Lyapunov exponent. Swift transitivity and contraction on large sets can be shown analogously to [13, Proposition 3.10] as the noise in our case is additive in the sense that every set of positive Lebesgue measure is reached with positive probability and the vector field obviously exhibits a monotonicity condition on large sets with respect to the y -variable. The disintegrations μ_ω of the ergodic invariant measure μ for the skew product flow Θ are Dirac measures in this case. It is clear that in the case of $\lambda_1 > 0$, the random attractor $\{A_\omega\}_{\omega \in \Omega}$ is not given by a random equilibrium. \square

The positive top Lyapunov exponent is the only characterization of chaos we can give in this case as an analysis in the sense of [29] seems not feasible for white noise. However, the geometric mechanism of shear-induced chaos can still be understood along the same lines: the white noise drives some points on the limit cycle up and some down. Due to the phase amplitude coupling b , the points with larger y -coordinates move faster in the ϑ -direction. At the same time, the dissipation force with strength α attracts the curve back to the limit cycles. This provides a mechanism for stretching and folding characteristic of chaos. The transition to chaos in the continuous time stochastic forcing is much faster than in the case of periodic kicks due to the effect of large deviations [20]. This is due to the fact that points end up in areas with arbitrarily large values of y with positive probability. Hence, not so much shear is needed to generate the described stretching and folding due to phase amplitude coupling. However, for very small shear and noise, the dissipation leads to sinks being formed between these large deviation events, and the attractor ends up to be a singleton.

In Figure 2, we show the top Lyapunov exponent as a function of σ for fixed b and α according to formula (2.14). We have used numerical integration up to machine precision to calculate λ_1 . The bifurcations of the sign of λ_1 at $\sigma_0(\alpha, b)$ as clearly seen in Figures 2a-2c. Furthermore, note that $\lambda_1 \rightarrow 0$ from below for $\sigma \rightarrow 0$. The explorations suggest that $\sigma_-(\alpha, b) = \sigma_0(\alpha, b) = \sigma_+(\alpha, b)$, where $\sigma_0(\alpha, b)$ is an increasing function of α and a decreasing function of b , or differently phrased: the larger the proportion of shear to dissipation, b/α , the smaller the bifurcation point $\sigma_0(\alpha, b)$. In Figure 2d, we choose small values of b and α , but b/α large. We see no negative values of λ_1 as we would have to take values of σ too small for the numerical integration.

We have chosen the same parameter regimes as in [20], where Lin and Young investigate numerically the Lyapunov exponents of the system given by

$$\begin{aligned} dy &= -\alpha y dt + \sigma \sin(2\pi\vartheta) \circ dW_t, \\ d\vartheta &= (1 + by) dt, \end{aligned} \quad (3.3)$$

taking $\vartheta \in [0, 2\pi]$. Their numerical results show exactly the same qualitative behaviour apart from a slightly different scaling due to the factor 2π . We conclude that the simpler choice of the diffusion coefficient in our case does not change the qualitative behaviour. This is not a surprise, since we can derive formula (2.14) for λ_1 if we choose f to be piecewise linear on the intervals $[\frac{i}{4}, \frac{i+1}{4}]$ for $i = 0, 1, 2, 3$ with $|f'|$ constant such that it represents a linear approximation of the sin-function.

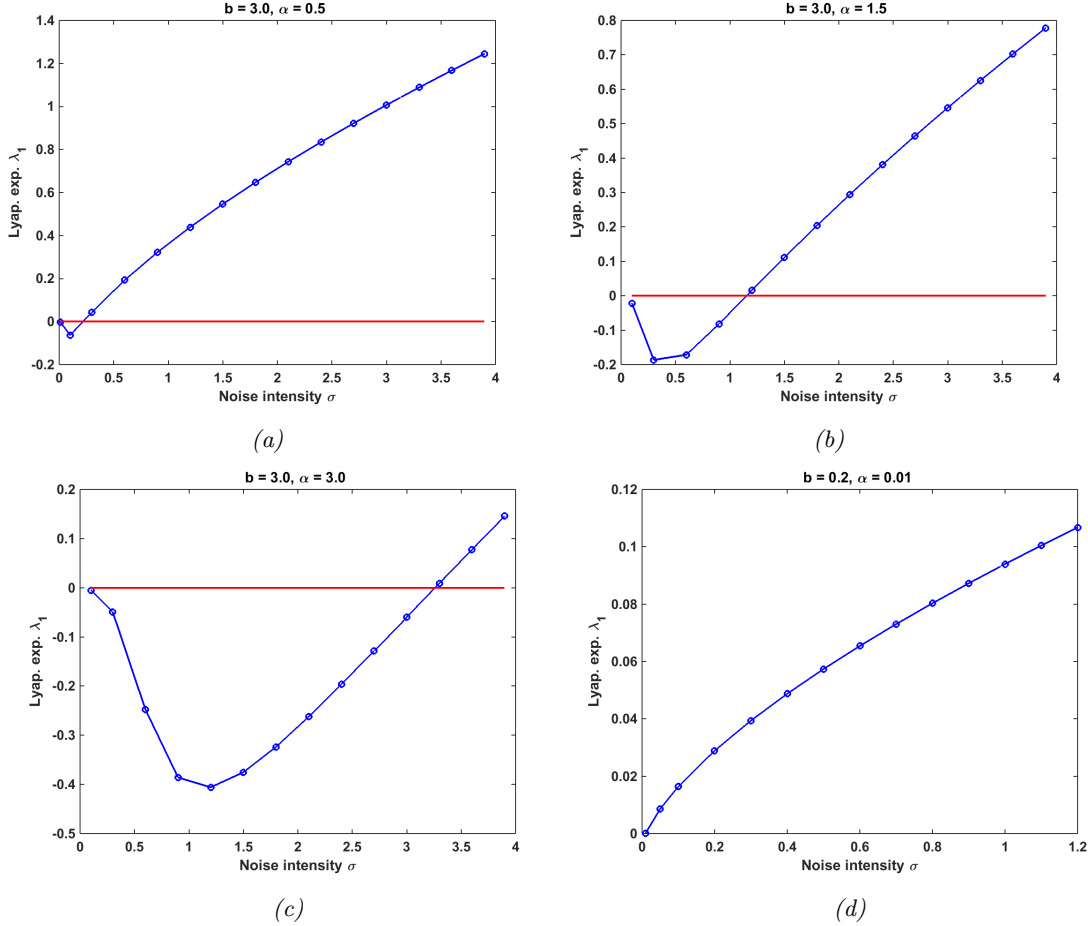


Figure 2: The top Lyapunov exponent λ_1 as a function of σ for fixed b and α . The dots indicate the values of λ_1 that were calculated according to (2.14) using numerical integration. Figures 2a-2c illustrate that $\sigma_0(\alpha, b)$ increases monotonously in α . In Figure 2d, b and α are small, but b/α is large. We don't see the transition to $\lambda_1 < 0$ since we would have to take values of σ too small for the numerical integration.

Furthermore, we confirm Lin and Young's conjecture concerning a scaling property of λ_1 with respect to the parameters. For model (3.3) they observed numerically that, under the transformations $\alpha \mapsto k\alpha$, $b \mapsto kb$ and $\sigma \mapsto \sqrt{k}\sigma$, λ_1 transforms approximately as $\lambda_1 \mapsto k\lambda_1$. This scaling property holds exactly for our model (1.1).

Proposition 3.3. *Consider the stochastic differential equation (1.1), where the function f is of the form (1.2). Then the top Lyapunov exponent λ_1 as given by (2.14) satisfies*

$$\lambda_1(k\alpha, kb, \sqrt{k}\sigma) = k\lambda_1(\alpha, b, \sigma) \quad \text{for all } k \in \mathbb{R}^+ \setminus \{0\}. \quad (3.4)$$

Proof. First note that by the change of variables $v \mapsto \frac{v}{k^2}$, and $u \mapsto \frac{u}{k^2}$ respectively, we obtain

$$\int_0^\infty v m_{\sqrt{k}\sigma, kb, k\alpha}(v) dv = \frac{\int_0^\infty v \frac{1}{\sqrt{v}} \exp\left(-\frac{\sigma^4 b^4}{6} k^6 v^3 + \frac{\alpha^2}{2} k^2 v\right) dv}{\int_0^\infty \frac{1}{\sqrt{u}} \exp\left(-\frac{\sigma^4 b^4}{6} k^6 u^3 + \frac{\alpha^2}{2} k^2 u\right) du} = \frac{1}{k^2} \int_0^\infty v m_{\sigma, b, \alpha}(v) dv.$$

Hence, we can conclude that

$$\lambda_1(\sqrt{k}\sigma, kb, k\alpha) = -k \frac{\alpha}{2} + k^3 \frac{b^2 \sigma^2}{2} \frac{1}{k^2} \int_0^\infty v m_{\sigma, b, \alpha}(v) dv = k \lambda_1(\sigma, b, \alpha).$$

□

3.2 Robustness of the result

We can perturb f around the point of non-differentiability at $\frac{1}{2}$ and show that the sign of λ_1 is preserved under small perturbations. This demonstrates the robustness of the bifurcation behaviour. In more detail: we choose $f_\varepsilon = f$ on $[0, \frac{1}{2} - \varepsilon] \cup [\frac{1}{2} + \varepsilon, 1]$ but smooth on the whole interval with $|f_\varepsilon|, |f'_\varepsilon| \leq 1$ such that $f_\varepsilon \rightarrow f$ uniformly. The existence of such f_ε follows from standard analysis. The corresponding model for f_ε is given by

$$\begin{aligned} dy &= -\alpha y dt + \sigma f_\varepsilon(\vartheta) \circ dW_t, \\ d\vartheta &= (1 + by) dt. \end{aligned} \tag{3.5}$$

Let λ_1^ε denote the top Lyapunov exponent for the random dynamical system $(\theta, \varphi_\varepsilon)$ induced by (3.5). Then indeed $\lambda_1^\varepsilon \rightarrow \lambda_1$ as $\varepsilon \rightarrow 0$, which we sketch in the following: denote by $\tilde{\rho}_\varepsilon$ the stationary density for the process in (y, ϑ, ϕ) where ϕ is given as in (2.7) for f_ε , carrying the information from the linearisation along trajectories:

$$d\phi = (\alpha \cos \phi \sin \phi + b \cos^2 \phi) dt - \sigma f'_\varepsilon(\vartheta) \sin^2 \phi \circ dW_t.$$

Existence and uniqueness of $\tilde{\rho}_\varepsilon$ are guaranteed by similar considerations as in Section 2. We denote the corresponding Fokker–Planck operators (cf. Appendix) for the systems in (y, ϑ, ϕ) with the diffusion coefficients f and f_ε by \mathcal{L}^* and $\mathcal{L}_\varepsilon^*$ respectively. The stationary Fokker–Planck equations read $\mathcal{L}_\varepsilon^* \tilde{\rho}_\varepsilon = 0$ and $\mathcal{L}^* \tilde{\rho} = 0$ where $\tilde{\rho}$ denotes the stationary density for the system associated to f . Note that we use the symbols $\tilde{\rho}$ and $\tilde{\rho}_\varepsilon$ both for the measures and corresponding densities.

In the following, we denote by $D = \mathbb{R} \times [0, 1) \times [0, \pi)$ the domain of (y, ϑ, ϕ) . Recall from (2.6) that

$$\lambda_1 = \int_D q(\phi) \tilde{\rho}(d\phi, d\vartheta, dy), \quad \lambda_1^\varepsilon = \int_D q_\varepsilon(\phi, \vartheta) \tilde{\rho}_\varepsilon(d\phi, d\vartheta, dy),$$

where

$$\begin{aligned} q(\phi) &= -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2} \sigma^2 \sin^2 \phi (1 - 2 \cos^2 \phi), \\ q_\varepsilon(\phi, \vartheta) &= -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2} \sigma^2 f'_\varepsilon(\vartheta)^2 \sin^2 \phi (1 - 2 \cos^2 \phi). \end{aligned}$$

We define

$$\lambda_1^{\varepsilon, \varepsilon} := \int_D q(\phi) \tilde{\rho}_\varepsilon(y, \vartheta, \phi) d\phi d\vartheta dy.$$

Observe that we obtain the following estimate for small enough $\varepsilon > 0$:

$$|\lambda_1 - \lambda_1^\varepsilon| \leq |\lambda_1 - \lambda_1^{\varepsilon, \varepsilon}| + |\lambda_1^{\varepsilon, \varepsilon} - \lambda_1^\varepsilon| \leq C_1(\alpha, b, \sigma) \|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} + \varepsilon C_2(\sigma) \max_D |\tilde{\rho}_\varepsilon(y, \vartheta, \phi)|. \tag{3.6}$$

The densities $\tilde{\rho}_\varepsilon$ are obviously uniformly bounded in ϑ and ϕ . Furthermore, since $\sigma f_\varepsilon(\vartheta) \leq \sigma$, the decay of $\tilde{\rho}_\varepsilon$ in y is at least the same as for the stationary density of the Ornstein–Uhlenbeck process

$$dy = -\alpha y dt + \sigma dW_t,$$

which is given by

$$p(y) = \frac{1}{Z} \exp\left(-\frac{\alpha}{2\sigma^2} y^2\right), \quad (3.7)$$

where Z is the normalisation constant. We deduce that there exists a constant $C > 0$ such that $\max_D |\tilde{\rho}_\varepsilon(y, \vartheta, \phi)| < C$ for any small $\varepsilon > 0$, which implies that the second term on the right-hand side of (3.6) vanishes as $\varepsilon \rightarrow 0$.

Hence, it remains to show that $\|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} \leq K\varepsilon$ for some constant $K > 0$. Since all the coefficients are bounded, we have for any sufficiently regular $\psi : D \rightarrow \mathbb{R}$,

$$\begin{aligned} & |\mathcal{L}_\varepsilon^* \psi(y, \vartheta, \phi) - \mathcal{L}^* \psi(y, \vartheta, \phi)| \\ & \leq C_3(\alpha, b, \sigma) \mathbf{1}_{[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]}(\vartheta) \left(\left| \frac{\partial}{\partial \phi} \psi(y, \vartheta, \phi) \right| + \left| \frac{\partial^2}{\partial^2 \phi} \psi(y, \vartheta, \phi) \right| + \left| \frac{\partial^2}{\partial^2 y} \psi(y, \vartheta, \phi) \right| + \left| \frac{\partial^2}{\partial \phi \partial y} \psi(y, \vartheta, \phi) \right| \right). \end{aligned} \quad (3.8)$$

We consider the problem in the weighted function spaces $W_p^{2,2}(D)$ to $L_p^2(D)$ where p is given by (3.7). We understand \mathcal{L}^* and $\mathcal{L}_\varepsilon^*$ as linear bounded Fredholm operators from $W_p^{2,2}(D)$ to $L_p^2(D)$ (as they are weakly elliptic from a weighted Hilbert space to a weighted Hilbert space). Since $W_p^{2,2}$ is a Hilbert space, we can define the projection P to $\ker(\mathcal{L}^*)$ which is a one-dimensional and hence closed subspace. We can further define the linear operator $\mathcal{S} : \mathcal{L}^*(W_p^{2,2}(D)) \rightarrow W_p^{2,2}(D)$ as the pseudoinverse of \mathcal{L}^* , i.e. define it via $\mathcal{S}\mathcal{L}^* = \text{Id} - P$. Since the range of a Fredholm operator is closed, \mathcal{S} is a linear and bounded operator by the open mapping theorem. Now choose $\tilde{\rho}_\varepsilon \in \ker(L_\varepsilon^*)$ such that $P\tilde{\rho}_\varepsilon = \tilde{\rho}$ (there is a normalisation error that can be easily resolved in the limit as $\varepsilon \rightarrow 0$). We use

$$\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon) = \mathcal{L}_\varepsilon^* \tilde{\rho}_\varepsilon - \mathcal{L}^* \tilde{\rho}_\varepsilon,$$

and (3.8), in combination with the uniform boundedness and decay of $\tilde{\rho}_\varepsilon$ and its derivatives in all variables, to deduce

$$\begin{aligned} \|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{L^1(D)} & \leq C_4 \|\tilde{\rho} - \tilde{\rho}_\varepsilon\|_{W_p^{2,2}(D)} = C_4 \|\mathcal{S}\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon)\|_{W_p^{2,2}(D)} \\ & \leq C_4 \|\mathcal{L}^*(\tilde{\rho} - \tilde{\rho}_\varepsilon)\|_{L_p^2(D)} \|\mathcal{S}\|_{B(\mathcal{L}^*(W_p^{2,2}(D)), W_p^{2,2}(D))} \leq K\varepsilon, \end{aligned}$$

for some constant $K > 0$.

4 Summary and Outlook

We have investigated systems with limit cycles on a cylinder perturbed by white noise. We were able to show a transition from negative to positive top Lyapunov exponents for fixed dissipation parameter α and big enough noise σ and/or shear b . This implies a bifurcation of the random attractor from a random equilibrium to random strange attractor.

In the case of positive Lyapunov exponents, it remains an open problem to describe the attractor using concepts from ergodic theory, as entropy and SRB measures [18, 29], in order to have a more rigorous notion of chaos.

The results of this paper may well be relevant to shed more light on the problem of stochastic Hopf bifurcation, where numerical studies indicate a transition from negative to positive Lyapunov exponent as explained in the Introduction.

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Appendix

Invariant measures

A random dynamical system (θ, φ) is associated with a skew product flow Θ on $\Omega \times X$ given by

$$\Theta_t(\omega, x) = (\theta_t \omega, \varphi(t, \omega)x).$$

The notion of an invariant measure for the random dynamical system is given as follows [3, Definition 2].

Definition 4.1. A probability measure μ on $\Omega \times X$ is invariant for the random dynamical system (θ, φ) if

- (i) $\Theta_t \mu = \mu$ for all $t \in \mathbb{R}_0^+$,
- (ii) the marginal of μ on Ω is \mathbb{P} , i.e. μ can be factorised uniquely into $\mu(d\omega, dx) = \mu_\omega(dx) \mathbb{P}(d\omega)$ where $\omega \mapsto \mu_\omega$ is a random measure on X .

The marginal of μ on the probability space is demanded to be \mathbb{P} as we assume the model of the noise to be fixed. Note that the invariance of μ reads in terms of the random measure on the state space

$$\varphi(t, \omega) \mu_\omega = \mu_{\theta_t \omega} \quad \mathbb{P}\text{-a.s. for all } t \in \mathbb{R}. \quad (4.1)$$

For a random dynamical system (θ, φ) induced by a stochastic differential equation, there is a one-to-one correspondence between invariant measures ρ for the associated Markov semigroup and $\mathcal{F}_{-\infty}^0$ -measurable random measures $\omega \mapsto \mu_\omega$ which are invariant in the sense of (4.1) [9]: If $\omega \mapsto \mu_\omega$ is a $\mathcal{F}_{-\infty}^0$ -measurable invariant random measure, then

$$\rho(\cdot) = \mathbb{E} \mu(\cdot) = \int_{\Omega} \mu_\omega(\cdot) \mathbb{P}(d\omega)$$

is an invariant measure for the Markov semigroup. If ρ is an invariant measure for the Markov semigroup, then

$$\mu_\omega = \lim_{t \rightarrow \infty} \varphi(t, \theta_{-t}\omega)\rho$$

exists \mathbb{P} -a.s. and is an $\mathcal{F}_{-\infty}^0$ -measurable invariant random measure.

Random attractors

By random attractor we mean a random pullback attractor as in [13]:

Definition 4.2. A random pullback attractor is a set valued map $A : \Omega \rightarrow \mathcal{P}(X)$ which is \mathbb{P} -a.s. compact, satisfies $\omega \mapsto d(x, A(\omega))$ being \mathcal{F} -measurable for each $x \in X$ and the following two properties:

(i) $\{A(\omega)\}_{\omega \in \Omega}$ is φ -invariant, i.e.

$$\varphi(t, \omega)A(\omega) = A(\theta_t \omega)$$

for all $t \geq 0$ and almost all $\omega \in \Omega$,

(ii) for every compact $B \subset X$, we have \mathbb{P} -a.s.

$$\lim_{t \rightarrow \infty} \sup_{x \in B} d(\varphi(t, \theta_{-t}\omega)x, A(\omega)) = 0.$$

$\{A(\omega)\}_{\omega \in \Omega}$ is called a *weak attractor* if it satisfies the latter property with almost sure convergence replaced by convergence in probability.

Multidimensional conversion formula from Stratonovich to Itô integral

Consider the Stratonovich SDE

$$dX_t = \overline{F}(t, X_t)dt + G(t, X_t) \circ dW_t,$$

where $\overline{F} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called the drift of the SDE, $G : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ the diffusion of the SDE and W_t is an m -dimensional Wiener process. In accordance with [12], the equation has the same solutions as the Itô SDE

$$dX_t = F(t, X_t)dt + G(t, X_t)dW_t,$$

where

$$\overline{F}_i(t, X_t) = F_i(t, X_t) - \frac{1}{2} \sum_{j=1}^d \sum_{k=1}^m G_{jk}(t, X_t) \frac{\partial G_{ik}}{\partial X_j}(t, X_t), \quad i = 1, \dots, d.$$

The Fokker–Planck equation

Consider the Itô SDE

$$dX_t = F(t, X_t)dt + G(t, X_t)dW_t,$$

where $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is called the drift of the SDE and $G : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ the diffusion of the SDE and W_t is an m -dimensional Wiener process. The so called Fokker–Planck equation describes the evolution of the density of the process X_t under sufficient (classical or Sobolev) regularity of the coefficients:

$$\frac{\partial p(t, x)}{\partial t} = - \sum_{i=1}^d \frac{\partial}{\partial x_i} [F_i(t, x)p(x, t)] + \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j} [D_{ij}(x, t)p(x, t)],$$

with diffusion tensor

$$D_{ij}(x, t) = \sum_{k=1}^m G_{ik}(x, t)G_{jk}(x, t).$$